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A VARIATIONAL PRINCIPLE AND THE CONVERGENCE OF A FINITE-ELEMENT METHOD BASED ON ASSUMED STRESS DISTRIBUTION*

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Abstract—A variational principle is formulated as the foundation of the finite-element method proposed by Pian. Through this formulation an extension of Pian's original proposal is made and the convergence of the approximate solution to the exact one is proved.

1. INTRODUCTION

IN ORDER to guarantee the monotonic convergence of the solution given by the direct stiffness method for the finite element analysis of elastic continuum, it has been shown [1] that the assumed displacement functions for the derivation of the stiffness matrices of the individual elements must satisfy the compatibility conditions at the interelement boundaries. In general, for elements of irregular geometry, there is no assurance that such compatible functions can be constructed. In the case of the bending of plates of triangular or quadrilateral planforms [2, 3] the compatible displacement functions must be derived from complicated processes and the resulting functional forms are also very complicated. Pian [4] has proposed a method which, in principle, can provide compatible stiffness matrices for elements of any geometry. In Pian's formulation of the finite element method, the displacements (and, in the case of plates [5, 6] or shells, the displacements and the normal slopes) are assumed only over the interelement boundaries, and stresses are assumed in the interior of each individual element. The boundary displacements are so assumed that compatibility with the neighboring elements is maintained while the assumed stresses satisfy the equilibrium equations. It is a comparatively easier task to construct displacement functions only along the boundary than to develop a continuous displacement function over the entire volume to maintain the compatibility with the neighboring elements.

The present paper is to show that such a method is actually an application of a variational principle and to prove its convergence. A slight extension of Pian's original proposal will also be given in this paper.

2. A VARIATIONAL PRINCIPLE FOR THREE-DIMENSIONAL PROBLEMS

Let us define a functional, Π

$$\Pi = \sum_n \left[\int_{\partial V_n} T_i u_i \, dS - \int_{S_{\sigma_n}} (T_i)_0 u_i \, dS - U_n(\sigma_{ij}) \right]. \tag{1}$$

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U_n is the strain energy of a subregion V_n of the region V being considered, and is expressed in terms of stresses. $V = \sum_n V_n$ is the total volume being considered. ∂V_n is the boundary of V_n and $\partial V_n = S_{\sigma_n} + S_{u_n} + S_n$ with S_{σ_n}, S_{u_n}, and S_n being the portion of ∂V_n where surface traction T_i, displacement u_i, and neither traction nor displacement, respectively, are prescribed. Subscript 0 denotes a prescribed quantity. The surface traction T_i is equal to $\nu_j \sigma_{ij}$, where ν_j is the component of the surface normal. The following theorem holds for the functional Π:

Theorem. Of all the admissible stresses and displacements, the actual solution i.e. the one which satisfies the compatibility conditions in all V_n and makes all V_n in equilibrium with each other, is distinguished by the stationary value of Π and the solution is unique. The stress is admissible if it is of class C^1 in each V_n and satisfies the equilibrium equations

$$\sigma_{ij,j} = F_i \tag{2}$$

in V_n. The displacement is admissible if it is continuous on ∂V_n and

$$u_i = (u_i)_0 \tag{3}$$

on S_{u_n}.

To prove this theorem, let us consider

$$U_n(\sigma_{ij}) = \tfrac{1}{2} \int_{V_n} K_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}V \tag{4}$$

where $K_{ijkl} = K_{jikl} = K_{klij}$ and is positive definite. The first variation of Π is

$$\begin{aligned} \delta\Pi = \sum_n \left[\int_{\partial V_n} \nu_j u_i \, \delta\sigma_{ij} \, \mathrm{d}S - \int_{V_n} K_{ijkl} \sigma_{ij} \, \delta\sigma_{kl} \, \mathrm{d}V \right] \\ + \sum_n \int_{S_n} \nu_j \sigma_{ij} \, \delta u_i \, \mathrm{d}S \\ + \sum_n \int_{S_{\sigma_n}} [\nu_j \sigma_{ij} - (T_i)_0] \, \delta u_i \, \mathrm{d}S. \end{aligned} \tag{5}$$

If

$$\delta\sigma_{ij,j} = 0 \qquad \text{in } V_n \tag{6}$$

the term in the first sum of equation (5) is actually the first variation of the complementary energy of the volume V_n for a given u_i on ∂V_n. Clearly, for the actual solution of the problem, we have $\delta\Pi = 0$. If $\delta\Pi = 0$, for all admissible $\delta\sigma_{ij}$ and δu_i, each of the sums of equation (5) must vanish. From the first sum we obtain the compatibility equations in V_n and a strain–stress relation on ∂V_n. From the second and third sum we require the equilibrium between the adjacent intersurface of the elements and the satisfaction of the stress boundary conditions, respectively. The proof of uniqueness is standard by using the property of positive definiteness of U_n.

In the case of the plate theory, the form of Π must be appropriately modified as follows:

$$\Pi = \sum_n \left[\int_{\partial A_n} q_n w \, dl + \int_{\partial A_n} M^{\alpha\beta} \omega_{,\alpha} v_\beta \, dl - \int_{l\sigma_{n2}} (M_n)_0 \frac{\partial w}{\partial N} \, dl \right. \tag{7}$$
$$\left. - \int_{l\sigma_{n2}} \left(q_n + \frac{\partial}{\partial l} M^{\alpha\beta} v_\alpha \frac{\partial l}{\partial x^\beta} \right)_0 w \, dl - U_n(M^{\alpha\beta}) \right].$$

Here, $A = \sum_n A_n$ is the entire surface of the plate. ∂A_n is the boundary of A_n and $\partial A_n = l_{\sigma_{n1}} + l_{u_{n1}} + l_n$, $l_{\sigma_{n1}}$ being the portion of ∂A_n where the transverse shear, q_n, and twisting moments M_{ns} are prescribed, $l_{\sigma_{n2}}$, where normal moment M_n is prescribed, $l_{u_{n1}}$, where the lateral displacement w is prescribed, $l_{u_{n2}}$, where the normal slope, $\partial w/\partial n$, is prescribed and l_n, where neither boundary loading nor displacement is prescribed. Other symbols in this expression are: $M^{\alpha\beta}$, the moment resultants, v, the direction cosine, $U_n(M^{\alpha\beta})$, the bending energy of the element A_n. The normal moment M_n is defined by the equation

$$M_n = M^{\alpha\beta} v_\alpha v_\beta. \tag{8}$$

The displacement w and $\partial w/\partial n$ must be equal to the prescribed values on $l_{u_{n1}}$ and $l_{u_{n2}}$ respectively.

3. APPLICATION OF THE VARIATIONAL PRINCIPLES

We can show that Pian's scheme is a direct application of the theorem in the previous section. Here we shall give a brief derivation of Pian's scheme and make a slight extension of it by including the body force. For simplicity, we will consider only the expression of Π given in equation (1).

Let us begin by expressing the stress distribution $\{\sigma\}$ in a subregion V_n in terms of m undetermined stress coefficients $\{\beta\}$ and a vector function $[P_B]\{\beta_B\}$ as

$$\{\sigma\} = [P]\{\beta\} + [P_B]\{\beta_B\} \tag{9}$$

where $[P_B]\{\beta_B\}$ is a particular solution of the equilibrium equations and vanishes if the body forces are zero. $[P]\{\beta\}$ are so chosen such that the homogeneous equations of equilibrium in V_n are satisfied. The surface traction T_j $(= v_i \sigma_{ij})$ and the displacement u_i on ∂V_n can be written in matrix form

$$\{\tau\} = [R]\{\beta\} + [R_B]\{\beta_B\} \tag{10}$$

$$\{u\} = [Q]\{q\} \tag{11}$$

where $[R_B]\{\beta_B\}$ corresponds to $[P_B]\{\beta_B\}$, $\{q\}$ are k undetermined coefficients (nodal displacement) and $[Q]\{q\}$ are so chosen that $\{u\}$ is continuous on all ∂V_n and equals the prescribed displacements on S_{u_n}. Then Π can be written as

$$\Pi = \sum_n (\{\beta\}^T[T]\{q\} + \{\beta_B\}^T[T_B]\{q\}$$
$$- \tfrac{1}{2}\{\beta\}^T[H]\{\beta\} - \{\beta\}^T[H_B]\{\beta_B\}$$
$$- \{S_0\}^T\{q\} - \tfrac{1}{2}B_n) \tag{12}$$

where

$$
\begin{aligned}
[T] &= \int_{\partial V_n} [R]^T[Q]\,\mathrm{d}S \\
[T_B] &= \int_{\partial V_n} [R_B]^T[Q]\,\mathrm{d}S \\
[H] &= \int_{V_n} [P]^T[N][P]\,\mathrm{d}V \\
[H_B] &= \int_{V_n} [P]^T[N][P_B]\,\mathrm{d}V \\
B_n &= \{\beta_B\}^T \int_{V_n} [P_B]^T[N][P_B]\,\mathrm{d}V\{\beta_B\} \\
\{S_0\}^T &= \int_{S_{\sigma n}} \{\tau_0\}^T[Q]\,\mathrm{d}S
\end{aligned}
\tag{13}
$$

and $[N]$ is defined by the equation

$$U_n = \tfrac{1}{2}\int_{V_n} \{\sigma\}^T[N]\{\sigma\}\,\mathrm{d}V. \tag{14}$$

The first variation of Π with respect to all β and all unspecified q is

$$\delta\Pi = \sum_n \{\delta\beta\}^T([T]\{q\}-[H_B]\{\beta_B\}-[H]\{\beta\})+\sum_n(\{\beta\}^T[T]+\{\beta_B\}^T[T_B]-\{S_0\}^T)\{\delta q\} \tag{15}$$

$\delta\Pi = 0$ implies

$$[H]\{\beta\} = [T]\{q\}-[H_B]\{\beta_B\} \tag{16}$$

and

$$\sum_n(\{\beta\}^T[T]+\{\beta_B\}^T[T_B]-\{S_0\}^T)\{\delta q\} = 0. \tag{17}$$

Substitution of equation (16) into equation (12) yields

$$\Pi = \sum_n \tfrac{1}{2}\{q\}^T[K]\{q\}-\sum_n \{q\}^T\{S\}+B \tag{18}$$

where

$$
\begin{aligned}
[K] &= [T]^T[H]^{-1}[T] \\
\{S\} &= \{S_0\}-[T_B]^T\{\beta_B\}+[T]^T[H]^{-1}[H_B]\{\beta_B\} \\
B &= \sum_n \tfrac{1}{2}(\{\beta_B\}^T[H_B]^T[H]^{-1}[H_B]\{\beta_B\}-B_n).
\end{aligned}
\tag{19}
$$

Matrix $[K]$ is identified as a stiffness matrix for the element V_n which is identically the same as that originally obtained by Pian [4]. Therefore, introduction of the body force does not change the stiffness matrix at all. The present extension, however, points out a way of lumping the body force systematically into the generalized forces.

Several important observations that should be made here are as follows:

(1) Although the matrix $[H]$ for each element is positive definite, the element stiffness matrix $[K]$ is $k\times k$ matrix and is positive semidefinite. Its rank is equal to min $(m, k-l)$,

where l is the number of degrees of freedom of the element corresponding to the rigid body motion.

(2) Let M and N be the total number of degrees of freedom in the assumed stresses and the assumed displacements, respectively, then M must be greater than N in order that a solution for $\delta\Pi = 0$ is obtainable. If $N \geq M$, equation (17) will give no solution for β's, in general. Even if there is a solution for β's in equation (17), the q's cannot be determined uniquely from equation (16) since $[T]$ is an $m \times k$ matrix and its rank is equal to $\min(m, k-l)$. If $m \geq k-l$ for each element, the existence of the solution for β's is guaranteed.

(3) In general, the finite element solution will depend upon the choice of the particular solution in equation (9). Let $\{\sigma_i\} = \{P\}\{\beta_i\}+\{P_{Bi}\}\{\beta_{Bi}\}$ and $\{q_i\}$, $i = 1, 2$ be two solutions corresponding to two different particular solutions. Let

$$\begin{aligned} \{q\} &= \{q_1\}-\{q_2\} \\ \{\sigma\} &= \{\sigma_1\}-\{\sigma_2\} \\ \{\beta\} &= \{\beta_1\}-\{\beta_2\} \\ [P_B]\{\beta_B\} &= [P_{B1}]\ \{\beta_{B1}\}-[P_{B2}]\{\beta_{B2}\}. \end{aligned} \tag{20}$$

Then $[P_B]\{\beta_B\}$ is a homogeneous solution of the equilibrium equation. The Π-functional for the difference $\{q\}$, $\{\sigma\}$ of the solutions are identical in form as equation (18) and satisfied

$$\sum \{\delta q\}^T([K]\{q\}-\{S\}) = 0 \tag{21}$$

with $\{S\}$ defined as equation (19), except that $\{S_0\} = 0$. If $[P]$ does not contain all the columns of $[P_B]$, $\{S\}$ will not be zero in general, i.e. $\{q\} \not\equiv 0$. But if $[P]$ contains all the columns of $[P_B]$ it can be easily seen that $[T_B]^T = [T]^T[H]^{-1}[H_B]$ in equations (19) and (13), thus $\{S\} = 0$. Therefore, we have $\{q\} = 0$, except the possibility of rigid body motion.

In practice, the elements of $[P]\{\beta\}$ are chosen to be polynomials. If the body force distribution is expressed in terms of a polynomial the particular solutions $[P_B]\{\beta_B\}$ are also polynomials. If we include in the polynomials in $[P]\{\beta\}$ all the terms of the different particular solutions $[P_B]\{\beta_B\}$ the finite element solution should be independent of the choice of these particular solutions.

Since the use of terms of a higher degree in $[P]\{\beta\}$ implies considerably more computation effort, it is of practical importance to approximate the body force distribution with only lower degree polynomials and to include the lowest degree terms for the particular solution.

(4) It is seen from the present variational principle that the stress boundary condition is not a restrained boundary condition; thus, in the approximate solution, it is not required to be satisfied identically by the assumed function. Therefore, Pian's suggestion [5] that the inclusion of a condition that the assumed stress must satisfy the prescribed surface traction may not be necessary. But, the numerical example on plate bending given in [6] indicates that for relatively large elements, a much more accurate result is obtained by making the stress satisfy the stress boundary conditions.

(5) This method involves a hybrid model. From the variational principle, it can be shown that the direct flexibility influence coefficient by this model is always bounded from below by that of a compatible model having the same type of interelement boundary displacements and bounded from above by that of an equilibrium model using the same type of interior stresses.

(6) Finally, it should be remarked that a dual variational principle to the one proposed in this paper has been suggested by Jones [7] and that the finite element displacement method developed by Yamamoto [8] can be interpreted as the application of this dual principle.

4. CONVERGENCE OF THE FINITE-ELEMENT SCHEME

The convergence proof is rather straightforward by following the method used in [1]. Suppose $\{\sigma\}$ and $\{u\}$ are the actual solution being considered and can be separated in the form

$$\{\sigma\} = \{\sigma_1\} + 0(\varepsilon) = [P]\{\alpha\} + [P_B]\{\beta_B\} + 0(\varepsilon) \tag{22}$$

in each subregion V_n and

$$\{u\} = \{u_1\} + 0(\varepsilon^2) = [Q]\{\gamma\} + 0(\varepsilon^2) \tag{23}$$

on ∂V_n where $[P]$, $[P_B]\{\beta_B\}$ and $[Q]$ are defined in equations (9) and (11) and ε is a measure of the size of the subregion V_n. If the slope of u is discontinuous at certain places, either the term $0(\varepsilon^2)$ should be changed to $0(\varepsilon)$ or properly choosing the element so that the discontinuity occurs on ∂V_n. Let $\{\sigma^*\}, \{u^*\}$ be the solution of equations (16) and (17). If $\bar{\Pi}$ and Π^* are used to denote the value of Π associated with $(\{\sigma\}, \{u\})$ and $(\{\sigma^*\}, \{u^*\})$, respectively, then we can show in the following that

$$|\Pi^* - \bar{\Pi}| = 0(\varepsilon) \quad \text{as} \quad \varepsilon \to 0. \tag{24}$$

Let

$$\begin{aligned} X &= \sum_n \left[2U_n(\sigma) - \int_{S_{\sigma_n}} (T_i)_0 u_i \, dS + \int_{S_{\sigma_n}} (T_i)_0 u_i^* \, dS \right] \\ Y &= \sum_n \left[2U_n(\sigma) - \int_{S_{u_n}} T_i (u_i)_0 \, dS + \int_{S_{u_n}} T_i^* (u_i)_0 \, dS \right]. \end{aligned} \tag{25}$$

Since $\{\sigma\}, \{u\}$ are the exact solution, we have, from equation (5),

$$\begin{aligned} X &= \sum_n \left[\int_{S_{u_n}} T_i (u_i)_0 \, dS + \int_{S_{\sigma_n}} (T_i)_0 u_i^* \, dS \right] \\ Y &= \sum_n \left[\int_{S_{\sigma_n}} (T_i)_0 u_i \, dS + \int_{S_{u_n}} T_i^* (u_i)_0 \, dS \right]. \end{aligned} \tag{26}$$

By equation (23) and the fact that u_i^* is equal to $(u_i)_1$ on S_{u_n}

$$(u_i)_0 - (u_i)^* = 0(\varepsilon^2)$$

and since

$$\sum_n \int_{S_n} T_i u_i^* \, dS = 0$$

we have

$$X = \sum_n \int_{\partial V_n} T_i u_i^* \, dS + 0(\varepsilon^2).$$

The term $0(\varepsilon^2)$ is identically zero if u_i^* equals $(u_i)_0$ on S_{u_n}. By equations (22) and (16), we can write

$$\begin{aligned} X &= \sum_n \int_{\partial V_n} (T_i)_1 u_i^* \, dS + 0(\varepsilon) \\ &= \sum_n \int_{V_n} K_{ijkl}\sigma_{ij}^*(\sigma_{kl})_1 \, dV + 0(\varepsilon) \\ &= \sum_n \int_{V_n} K_{ijkl}\sigma_{ij}^*\sigma_{kl} \, dV + 0(\varepsilon). \end{aligned} \tag{27}$$

From equation (23) Y of equation (26) can be written as

$$Y = \sum_n \left[\int_{S_{\sigma_n}} (u_i)_1 (T_i)_0 \, dS + \int_{S_{u_n}} (u_i)_1 T_i^* \, dS \right] + 0(\varepsilon^2).$$

We see that equation (17) may be expressed as

$$\sum_n \int_{S_{\sigma_n}} (u_i)^1 (T_i)_0 \, dS = \sum_n \left[\int_{S_{\sigma_n}} (u_i)_1 T_i^* \, dS + \int_{S_n} (u_i)_1 T_i^* \, dS \right]$$

by setting $[Q]\{\delta q\} = \{u\}_1$. Therefore

$$\begin{aligned} Y &= \sum_n \int_{\partial V_n} T_i^* (u_i)_1 \, dS + 0(\varepsilon^2) \\ &= \sum_n \int_{\partial V_n} T_i^* u_i \, dS + 0(\varepsilon^2) \\ &= \sum_n \int_{V_n} K_{ijkl}\sigma_{ij}^*\sigma_{kl} \, dV + 0(\varepsilon^2). \end{aligned} \tag{28}$$

From equations (1), (16), (17) and (25)

$$\begin{aligned} \Pi^* - \bar{\Pi} &= \sum_n [U_n(\sigma) + U_n(\sigma^*)] \\ &\quad - \sum_n \left[2U_n(\sigma) - \int_{S_{\sigma_n}} (T_i)_0 (u_i - u_i^*) \, dS \right] \\ &= \sum_n [U_n(\sigma) + U_n(\sigma^*)] - X \end{aligned}$$

or

$$\begin{aligned} \Pi^* - \bar{\Pi} &= \sum_n \left[2U_n(\sigma) - \int_{S_{u_n}} (u_i)_0 (T_i - T_i^*) \, dS \right] \\ &\quad - \sum_n [U_n(\sigma) + U_n(\sigma^*)] \\ &= Y - \sum_n [U_n(\sigma) + U_n(\sigma^*)]. \end{aligned}$$

From equations (5), (27) and (28), we get

$$\Pi^* - \bar{\Pi} = \sum_n U_n(\sigma - \sigma^*) + 0(\varepsilon)$$

or

$$\Pi^* - \bar{\Pi} = -\sum_n U_n(\sigma - \sigma^*) - O(\varepsilon^2)$$

which implies that

$$|\Pi^* - \bar{\Pi}| = O(\varepsilon).$$

The origin of the term $O(\varepsilon)$ in equation (27) is from the approximation in stresses. If we can use a better stress approximation [equation (9)] such that equation (22) can be written as

$$\{\sigma\} = [P]\{\alpha\} + [P_B]\{\beta_B\} + O(\varepsilon^2)$$

then it can easily be shown that

$$|\Pi^* - \bar{\Pi}| = O(\varepsilon^2). \tag{29}$$

Any further improvement of the stress approximation without improving the displacement approximation will not be able to change the approximation of Π^* to $\bar{\Pi}$ consistently, as in equation (29).

In equation (28), since $\Pi^* \to \bar{\Pi}$, we have

$$\sum_n U_n(\sigma - \sigma^*) \to 0. \tag{30}$$

Since the strain energy is positive definite, it can be concluded that $\{\sigma^*\}$ converges to $\{\sigma\}$ in the sense of equation (30), as ε tends to zero.

The convergence of $\{u^*\}$ can be proved as follows. Let

$$\{u^{**}\} = [A]\{q^*\}$$

be the exact displacement in V_n corresponding to the prescribed displacement $\{u^*\}$ $(= [Q]\{q^*\})$ on ∂V_n. The strain energy of the volume V associated with the displacement field $\{u^{**}\}$ is

$$\tfrac{1}{2}D(u^{**}) = \tfrac{1}{2}\sum_n \int_{V_n} C_{ijkl} e_{ij}^{**} e_{kl}^{**} \, dV \tag{31}$$

where $e_{ij}^{**} = \frac{1}{2}(u_{i,j}^{**} + u_{j,i}^{**})$. Since u^* must be a smooth function along the interelement boundaries as $\varepsilon \to 0$ (otherwise Π^* will be unbounded), the stresses given in equation (9) which satisfy equations (16) and (22) are approximately equal to $C_{ijkl}e_{kl}^{**}$. That is

$$\sigma_{ij}^* = C_{ijkl}e_{kl}^{**} + O(\varepsilon).$$

Therefore,

$$\tfrac{1}{2}D(u^{**}) = \sum_n U_n(\sigma^*) - O(\varepsilon). \tag{32}$$

It has been shown that [1]

$$\int_V (u - u^{**})^2 \, dV \le \frac{1}{\lambda} D(u - u^{**}) \tag{33}†$$

† In the case of $\sum \int_{S_{u_n}} dS = 0$, the stiffness matrix is positive semidefinite. One must then impose additional conditions of removing rigid body motion in order to prove this condition.

where $\lambda > 0$ and

$$\begin{aligned} D(u-u^{**}) &= D(u) - 2\int_V C_{ijkl}e_{ij}e_{kl}^{**}\,\mathrm{d}V + D(u^{**}) \\ &= 2\sum_n \left[U_n(\sigma) - \int_{\partial V_n} T_i u_i^*\,\mathrm{d}S + U_n(\sigma^*) \right] + D(u^{**}) - 2\sum_n U_n(\sigma^*) \\ &= 2\sum_n U_n(\sigma-\sigma^*) + D(u^{**}) - 2\sum_n U_n(\sigma^*). \end{aligned} \tag{34}$$

From equations (20) and (32), we have

$$D(u-u^{**}) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore

$$\int_V (u-u^{**})^2\,\mathrm{d}V \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

5. CONCLUSION

A functional is constructed for the foundation of the finite-element method proposed by Pian. The functional has an extremum associated with the actual solution of the problem, if the actual solution exists. By the application of the properties of the functional, the mean convergence of the approximate solution to the exact one is proved. From the theorem proposed, we can see that the stress boundary conditions are unrestrained boundary conditions and, in the dual method, the displacements are not restrained boundary conditions. In proving the theorem, we have also shown that in Pian's method the total number of degrees of freedom in stresses must be greater than or equal to that of the displacements and vice versa for the dual scheme in order to guarantee the existence of solutions. The solution will, in general, depend upon the choice of the particular solution of the equilibrium equation when the body forces are not zero. This difficulty can be avoided by making both the homogeneous and the inhomogeneous solutions to be polynomials of the same degree. By so doing, the inhomogeneous equilibrium equations may not be satisfied identically, but it will be within the same order of approximation of the finite-element method. This assumed stress method is a hybrid method. It can be shown that its solution will yield a structure more rigid than that by an equilibrium model using the same type of stress distribution within the elements, and more flexible than that by a compatible displacement model having the same interelement boundary displacements. From the proof of convergence, we can see that in order to obtain progressively better accuracy, we must improve the stress and the displacement approximations properly and simultaneously. For example, in applying Pian's method for the plane stress problem, if linear displacements are used for the boundary of a quadrilateral element, a constant or a linear stress distribution in the interior can be used. A linear distribution of stress will be the optimum choice. The inclusion of quadratic terms or higher order terms in stresses without the inclusion of the higher order terms of the displacements cannot guarantee an improvement in the solution.

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Абстракт—Формулируется вариационный принцип как основа метода конечного элемента предложенного Пианом. Вследствие такой формулировки развивается оригинальное предположение Пиана и исследуется сходимость приближенного решения с точным решением.